Ranking Urban Sustainability Performance Using the Colley Matrix: A National Comparative Analysis

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Introduction

Brief Overview on Linear Systems/Algebra:

What are Linear Systems?

- A collection of linear equations that describe relationships between variables.
- For example:

$$\begin{cases} 2x + 3y = 5 \\ 4x - y = 6 \end{cases}$$

- The goal: to find values of the variables that satisfy all equations simultaneously.

- How we accomplish this: **Linear Algebra**

Example

- First written example of linear systems appeared in China over 2,000 years ago

- Had to decide which of three types of rice (inferior, medium, and superior) yielded the most rice per cost

Example Continued

Variable	Rice type
x	Inferior rice
y	Medium rice
z	Superior rice

Example Continued

Yields can be represented by the following system of equations:

$$x + 2y + 3z = 39$$

$$x + 3y + 2z = 34$$

$$x + 2y + z = 26$$

Example Continued

The objective is to get the equations into the following form:

$$X = \ldots$$

$$Y = \ldots$$

$$Z = \ldots$$

Example Continued

$$x + 2y + 3z = 39$$
$$x + 3y + 2z = 34$$
$$x + 2y + z = 26$$

$\Rightarrow$

$$X = 2.75$$
$$Y = 4.25$$
$$Z = 9.25$$

Matrices and Vectors

- How else can we solve these equations?
  - Given a linear set of equations:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n &= b_2 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n &= b_m \end{aligned}$$

  - We can write the coefficient matrix of the system continuing the coefficients present of the left side of the standard form equations in the system. Similarly, we can construct an augmented matrix containing the constants from the original equations on the right side.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \qquad \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m \end{bmatrix}$$

Matrices and Vectors

Systems of equations can be solved by using matrices through elementary row operations (EROs)

Three types of Elementary Row Operations

1. Replacement: Replace one row by the sum of itself and a multiple of another row
2. Interchange: Swapping two rows
3. Scaling: Multiply all entries in a row by a nonzero constant

Matrices and Vectors

Using matrices, you can solve any system of equations given an invertible coefficient matrix A, and vector b. This is by using the equation $A^{-1}b = x$.

An invertible matrix is a square matrix that has an inverse (when multiplied $A^{-1}A = I$, where I is the identity matrix)

The identity matrix is defined as a square matrix with all diagonal values are 1, and the rest are 0.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To compute the inverse of a matrix row reduce the matrix [A I] to the form [I A^{-1}]. This can be done through elementary row operations.

Now, through matrix multiplication, you can multiply A^{-1} by vector b and obtain the solutions.

Methods

Colley's Matrix

Developed by Dr. Wesley Colley as a means of ranking sports teams

$$C\vec{r} = \vec{b}$$

$\vec{r}$: unknown ranking vector

$\vec{b}$: right hand side vector with entries

C: Colley's coefficient matrix

Colley's Matrix

Colley's Coefficient Matrix looks like this

$$C_{ij} = \begin{cases} 2 + t_i & i = j \\ -n_{ij} & i \neq j \end{cases} \implies C_{5\times 5} = \begin{bmatrix} 2+t_i & -n_{ij} & -n_{ij} & -n_{ij} & -n_{ij} \\ -n_{ij} & 2+t_i & -n_{ij} & -n_{ij} & -n_{ij} \\ -n_{ij} & -n_{ij} & 2+t_i & -n_{ij} & -n_{ij} \\ -n_{ij} & -n_{ij} & -n_{ij} & 2+t_i & -n_{ij} \\ -n_{ij} & -n_{ij} & -n_{ij} & -n_{ij} & 2+t_i \end{bmatrix}$$

Colley's Matrix

Colley's b vector looks like this

$$b_i = 1 + \frac{1}{2}(w_i - l_i) = \frac{2 + w_i - l_i}{2}$$

Colley's Matrix Example (in sports)

Team	Wins	Losses	Win %
Team A	1	2	33%
Team B	0	3	0%
Team C	3	0	100%
Team D	2	1	66%

Apply to Colley's Matrix (Cr = b):

C is the matrix:

$$\begin{bmatrix} 5 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 \\ -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 5 \end{bmatrix}$$

b is the vector:

$$\begin{bmatrix} 0.5 \\ -0.5 \\ 2.5 \\ 1.5 \end{bmatrix}$$

Calculate new rankings:

Team	Value	Rank
Team A	0.416	3
Team B	0.25	4
Team C	0.75	1
Team D	0.583	2

Team	Old Rank (Win %)	New Rank (Colleys)
Team A	3	3
Team B	4	4
Team C	1	1
Team D	2	2

Methods

Problem Statement: How can we fairly rank countries based on sustainability metrics?

Our Application

Why Colley's Matrix?

Background Summary of Waste Management Articles

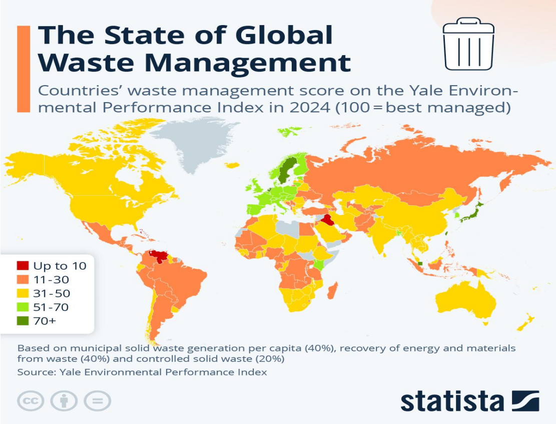
Core Summary by Agamuthu and Babel : In the previous 50 years, waste management in Asia has changed but still faces major challenges. Many factors including urbanization, population growth, developments in the economy have increased waste generation. Although technologies like waste-to-energy (WtE) and recycling systems are growing, many regions depend on open dumping and landfills.

Core Summary by Maalouf and Agamuthu: Over the past 50 years, waste management in less developed countries has changed from uncontrolled dumping to more civilized practices. However, systemic challenges—like lack of financial aid, lack of political will, poor leadership, and informal sector abuse—still impedes progress toward sustainable waste management.

More Detail of Separate Waste Management Articles

Summary by Almeida, Manzano, and Garcia:
The study performs a statistical analysis of 416 scientific papers (2007–2020) using the Scopus database to find global research on CE and waste management. It finds that research in this field has increased rather quickly, with a 94% increase in publications over the last five years, led mainly by Italy, Spain, the UK, China, Brazil, and India. Early research concentrated on industrial processes, but in recent years it has shifted towards waste management, recycling, sustainable development, and municipal solid waste, particularly in terms of COVID-19.

Summary of Study by Sakai, Yoshida, and Hirai:
This study aims to compare and analyze the development of 3R (Reduce, Reuse, Recycle) and waste management policies in the European Union, USA, Japan, Korea, and China. Its main purpose is to understand how different nations apply 3R strategies, how these policies contribute to resource conservation and greenhouse gas reduction, and which factors mainly cause policy changes—such as land scarcity, economic growth, or environmental pressures. The research also examines the integration of 3R with hazardous waste management, resource usage, and sustainable development to identify which practices are most efficient and which need improvement.



Key Metrics:

Metric	Description
% Renewable Electricity (2023)	% of electricity from renewable sources
Recycling	EPI score (1 to 100) for recycling, with 100 being full recycling and 0 being none.
Waste Generated Per Capita (2024)	Index score (1 to 100) for waste generation, with 100 being maximum waste generated and 0 none.
Wastewater Generated (2024)	Index score (1 to 100) for wastewater generation, with 100 being maximum wastewater and 0 none.
CO_2 Emissions (2023)	Carbon dioxide emissions in metric tons (Mt)
Pollution - AQI (2024)	Air Quality Index from 0 to 500, with 500 being extremely hazardous air and 0 being good, clean air

Extensions to Colley's Matrix

```
 B = 1+ 0.5 * (R*(Recycling - AvgsR) + E*(Renewables - AvgsE) + P*(-Pollution + AvgsP) +W*(-Wastewater + AvgsW) + C*(-Carbon + AvgsC));
```

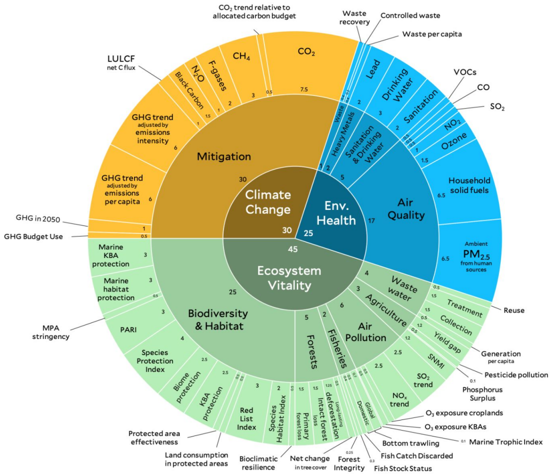
Inspiration from Evolutionary Principles

- Modeled the B-Matrix after evolutionary traits, reflecting how relative advantages improve survival chances
- Used the difference between a country's data and the average to measure how well or poorly it performs

Metric Weighting

- Standardized metrics to avoid large values (e.g., Carbon Emissions) from overpowering others
- Consulted the 2024 Yale EPI weighting chart, selecting a few relevant metrics
- Applied proportional weight to each metric based on its contribution to environmental impact

Data Explanation



Results

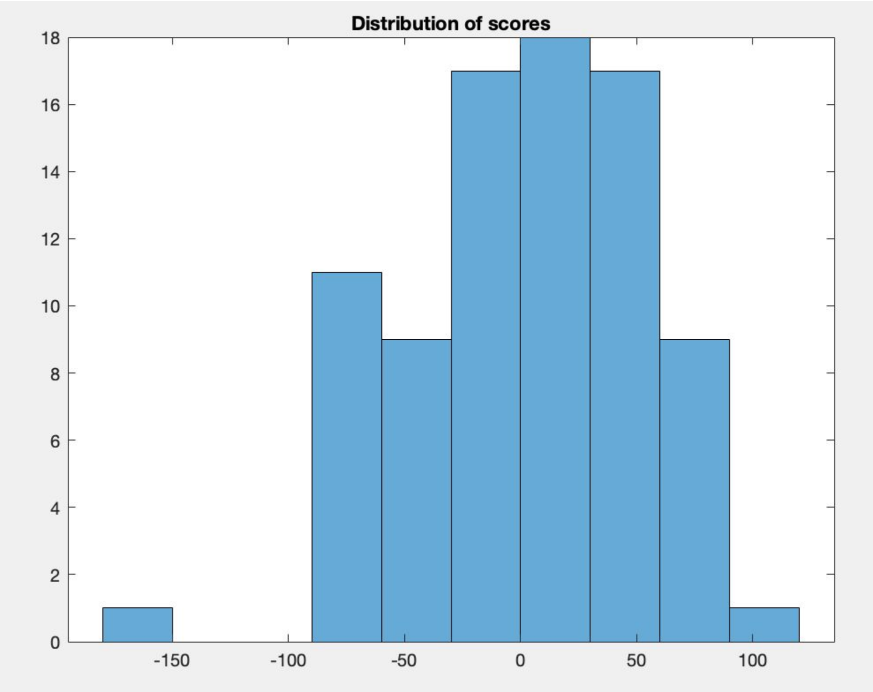
Results

- Here we alter the right hand b vector to depict environmental rankings
- Our left hand side will consist of the unknown r vector as well as the large C matrix, where we have each country “competing” against each other once
- An example is given below:

$$\underbrace{\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \text{United States} \\ \text{Canada} \end{bmatrix}}_{\vec{r}} = \underbrace{\begin{bmatrix} 1 + \frac{1}{2}\left(R * (Recycling_{US} - AvgsR) + \dots\right) \\ 1 + \frac{1}{2}\left(R * (Recycling_{CA} - AvgsR) + \dots\right) \end{bmatrix}}_{\vec{b}}$$

logarithms of population

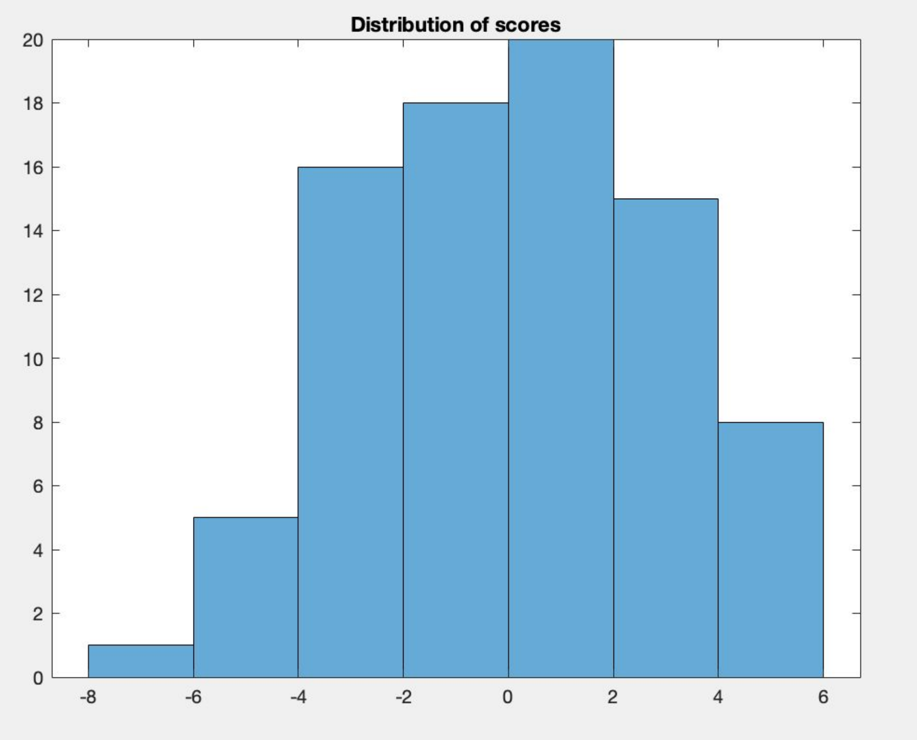
- by scaling our application of Colley's matrix for each metric by log(population), we account for population differences and rank accordingly.
- scaling by both log(population) and 1/(log(population)) generates the same graph but with slightly different country rankings:



	rank 1	2	3	4	5	79	80	81	82	83
log(pop.)	Australia	Norway	Finland	Canada	Germany	Bangladesh	Eritrea	Nepal	Cameroon	DR Congo
1/log(pop)	Australia	Germany	Canada	USA	UK	Nepal	Cameroon	Bangladesh	India	DR Congo

Per Capita

- modifying our original application from metric's distance from average to that metric's per capita value distance from mean per capita value helps proportion the population differences as well
- gives a graph that's slightly skewed left and similar but not identical results as the logarithms



	rank 1	2	3	4	5	79	80	81	82	83
per capita	Australia	Germany	Canada	Norway	Finland	Niger	Nepal	Cameroon	Eritrea	DR Congo

Weights Used for the Next Slides

R=5.303769237142165*1 %Recycling

E=235.6894053952179 %Renewables

P=1.780231813688434*17 %Polution

W=2.01179136725876*4 %Wastewater

C=0.2667792163659435*7.5 % Carbon

Top Ten Countries in Different Methods

	b=1+0.5*Weight*(Value-Avg(V))...	b=1+0.5*(+PosV-NegV..)	b=1+0.5*(+PosV-NegV …)*(1./log(Pop))	b=1+0.5*Weight*(Value-Avg(V))... *(1./log(Pop))	yale.edu method from countries chosen (Index)
#1	Finland: 12.7703	Finland:-78.1584	Finland: -3.7772	Finland: 2.2318	Germany: 74.5
#2	Sweden: 12.2643	Sweden: -78.1918	Sweden:-3.7786	New Zealand: 2.1966	Finland: 73.8
#3	New Zealand: 12.1622	Norway: -78.1940	Norway: -3.7795	Sweden: 2.1692	UK: 72.6
#4	Norway:11.3398	New Zealand: -78.2244	New Zealand:-3.7816	Norway: 2.1404	Sweden: 70.3
#5	Portugal: 10.1964	Namibia: -78.3539	Portugal:-3.7893	Portugal: 2.0417	Norway: 69.9
#6	Belarus: 9.4659	Nicaragua: -78.3550	Nicaragua:-3.7895	Belarus: 2.0013	Austria: 68.9
#7	Australia:9.4099	Portugal: -78.3658	Belarus: -3.7903	Namibia: 1.9886	France: 67.0
#8	Myanmar: 9.2566	Botswana: -78.3675	Bolivia: -3.7906	Australia: 1.9625	Poland: 64.2
#9	Bolivia: 8.7265	Belarus: -78.3775	Namibia:-3.7912	Bolivia: 1.9457	Spain: 64.0
#10	Namibia: 8.6360	Suriname: -78.3860	Botswana:-3.7925	Myanmar:1.9325	Australia: 63.1

Top Ten Lowest Countries in Different Methods

	b=1+0.5*Weight*(Value-Avg(V))...	b=1+0.5*(+PosV-NegV..)	b=1+0.5*(+PosV-NegV …)*(1./log(Pop))	b=1+0.5*Weight*(Value-Avg(V))... *(1./log(Pop))	yale.edu method from countries chosen (Index)
#74	Eritrea: -2.4821	Mexico: -81.1905	Mexico: -3.9364	Cameroon: 1.2750	Cambodia: 31.2
#75	Indonesia:-2.5815	Canada: -81.2711	Germany: -3.9476	Eritrea: 1.2445	Iraq: 30.3
#76	Nepal: -3.5388	Germany: -81.3187	Canada: -3.9532	Nepal: 1.2064	Madagascar: 30.1
#77	Iran: -7.3374	Saudi Arabia: -81.8744	Indonesia:-3.9846	Iran: 1.0128	Eritrea: 29.0
#78	Pakistan:-7.8483	Indonesia: -82.2653	Saudi Arabia :-3.9899	Pakistan: 1.0081	Bangladesh: 28.1
#79	Bangladesh: -11.2954	Iran: -83.0304	Iran: -4.0403	Bangladesh: 0.8181	India: 27.6
#80	DR Congo: -13.5099	Japan: -83.6519	Japan: -4.0689	DR Congo: 0.6822	Myanmar: 27.1
#81	India: -39.8781	India: -96.0086	India: -4.6189	India: -0.4756	Laos: 26.3
#82	USA: -43.0202	USA: -105.2215	USA: -5.1503	USA: -0.7744	Pakistan: 25.5
#83	China:-153.6287	China: -155.5641	China: -7.4451	China: -5.8738	Vietnam: 24.6

Conclusion

Conclusion

- Using different multipliers and b-vectors as applications of Colley's Matrix, we have produced many different viable rankings for different countries based on their environmental metrics.
    - metrics: percent renewable electricity, pollution AQI, recycling EPI, CO2 emissions, waste, and wastewater generated
- Overall all the different rankings produce similar results and trends, even with some variation.

Small summary of results (We have multiple valid rankings)

- While many countries' rankings stayed consistent regardless of 'b vector', some countries rank differed greatly among the different methods.

Future Extensions

Limitations to our model

- Omission of certain countries, limiting the scope of the model
- Focuses on only 6 factors, primarily environmental, which neglects other key aspects of urban sustainability

Assumptions we could loosen

- Colley’s original matrix was based off binary wins and losses. To apply that to something not binary like pollution & recycling metrics, we used a distance from the average. We could have perhaps tried other methods to simulate wins & losses, as the averages for certain metric aren’t necessarily base values.
- environmental sustainability is not a linear measurement

Other applications (Economic, social, etc.)

- Colley’s Matrix could be adapted to predict economic trends by incorporating factors like inflation, employment rate, and market behavior.
- It can also be applied in social contexts in the future

Questions?

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